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COMPACT HYPERSURFACES WITH CONSTANT SCALAR CURVATURE AND A CONGRUENCE THEOREM

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For an n-dimensional hypersurface M^n in the Euclidean space, we consider the rth mean curvature H_r, defined as the elementary symmetric polynomial of degree r in the principal curvatures of M^n. H_1, H_2, and H_n are the mean curvature, the scalar curvature, and the Gauss-Kronecker curvature respectively. The simplest global question concerning these geometric objects is the following:

> "*Given a compact hypersurface M^n embedded/immersed in the Euclidean space, such that H_r is constant for some $r = 1, \cdots, n$, is M^n a sphere?*"

The only solutions for this problem have been obtained in the cases $r = 1$ and $r = n$. If the mean curvature is constant and M^n is embedded, Aleksandrov [1] proved that M^n is a sphere. In the immersed case Hsiang, Teng, and Yu [3], and Wente [8], constructed nonspherical compact hypersurfaces in higher dimension and in $\mathbf{R}^3$, respectively. If the Gauss-Kronecker curvature is constant, then we conclude via the Hadamard theorem that M^n is strictly convex. But if M^n is strictly convex we know that $H_r =$ const., for some r, implies that M^n is a sphere (see Hsiung [4]). If $n = 2$ we obtain a classical result of Liebmann [6].

For the scalar curvature the problem has a special interest, which was proposed by Yau in [9].

In this paper we first prove that

> "*The sphere is the only compact hypersurface with constant scalar curvature embedded in the Euclidean space.*"

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This result follows from a modification of Reilly's proof for the Aleksandrov theorem [7]. It is an agreeable fact to observe that the ideas of these authors, in both proofs, can be adapted to several other contexts.

Our second result follows also from the Reilly method, and it concerns the extrinsic rigidity of compact hypersurfaces with nonnegative mean curvature. A classical result of Schur (see [2, p. 36]) can be stated, in the case of closed curves, as follows:

> "*Let C be a convex closed curve in the plane, and C' a closed curve in $\mathbf{R}^3$ of the same length as C. Let s be the arc parameter in both curves and $k(s)$ and $k'(s)$ the curvature of C and C', respectively. If $k'(s) \leqslant k(s)$, then C' is congruent to C.*"

The above result also follows easily from the Fenchel estimate for the total curvature of a closed curve. We will prove that the theorem extends to hypersurfaces. More precisely

> "*Let $\psi: M^n \to \mathbf{R}^{n+1}$ be an embedded compact hypersurface with nonnegative mean curvature H. Let $\psi': M^n \to \mathbf{R}^{n+m}$ be another isometric immersion with mean curvature vector H'. If $|H'| \leqslant H$, then both immersions are congruent.*"

Even more, the fact that ψ' is an isometric immersion is used in a very weak form: we need only assume that ψ' satisfies a certain integral inequality. We remark that Lawson and Tribuzy [5] have obtained a congruence result for compact surfaces in $\mathbf{R}^3$, assuming that $H = H'$. Clearly, if $n \geqslant 2$ the hypothesis $H \geqslant 0$ is weaker than convexity. For surfaces in $\mathbf{R}^3$ our result is applied, for instance, to some revolution tori.

1. Preliminaries

In this section we review some standard facts about the geometry of compact hypersurfaces. Let $\psi: M^n \to \mathbf{R}^{n+1}$ be an n-dimensional compact hypersurface embedded in the Euclidean space $\mathbf{R}^{n+1}$. Then M^n is the boundary of a compact domain $\Omega \subset \mathbf{R}^{n+1}$, $\partial\Omega = M^n$. Let N be the *interior* normal field of M, and $\{e_i\}$ an orthonormal basis in the tangent space of M. We denote by σ the second fundamental form with respect to the normal N, i.e., $\sigma(e_i, e_j) = -\langle N_*(e_i), e_j\rangle$, and let $H = \frac{1}{n}\Sigma_i \sigma(e_i, e_i)$ and S be the mean curvature of the immersion and the scalar curvature of M, respectively. From the Gauss equation we have

$$S = n^2H^2 - |\sigma|^2, \tag{1}$$

and by the Schwarz inequality

$$S \leqslant n(n-1)H^2, \tag{2}$$

the equality holding only at umbilical points. We will denote by dV and dA the canonical measures on $\mathbf{R}^{n+1}$ and M, respectively. Finally we denote by V and A the volume of Ω and the area of M respectively.

If we compute the Laplacian of the function $|x|^2$ on $\mathbf{R}^{n+1}$, x being the position vector field, we obtain $\bar{\Delta}|x|^2 = 2(n+1)$. So from the divergence theorem we have

$$(n+1)V + \int_M \langle \psi, N\rangle\, dA = 0. \tag{3}$$

We consider now the 1-form α on M defined by

$$\alpha(e_i) = \sum_j \left[\sigma(e_i, e_j)\langle \psi, e_j\rangle - \sigma(e_j, e_j)\langle \psi, e_i\rangle\right].$$

By direct computation we obtain that the divergence of α is given by

$$\operatorname{div}\alpha = \sum_i (\nabla\alpha)(e_i, e_i) = (n - n^2)H + \left(|\sigma|^2 - n^2H^2\right)\langle \psi, N\rangle.$$

Integrating on M and using (1) we get

$$n(n-1)\int_M H\, dA + \int_M S\langle \psi, N\rangle\, dA = 0. \tag{4}$$

Relation (4) is usually known as the second Minkowski formula. For the general case see Hsiung [4].

Given $f \in C^\infty(\bar{\Omega})$, we denote $z = f_{|M}$ and $u = \partial f/\partial N$. So z and u are smooth functions on M. Reilly's formula [7] states that

$$\int_\Omega \left[(\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2\right] dV = \int_M \left[-2(\Delta z)u + nHu^2 + \sigma(\nabla z, \nabla z)\right] dA, \tag{5}$$

where $\bar{\Delta} f$ and $\bar{\nabla}^2 f$ are the Laplacian and the Hessian of f in $\mathbf{R}^{n+1}$, and ∇z and Δz are the gradient and the Laplacian of z in M.

2. Hypersurfaces with constant scalar curvature

Theorem 1. *Let M^n be an n-dimensional compact hypersurface embedded in the Euclidean space $\mathbf{R}^{n+1}$. If the scalar curvature of M^n is constant, then M^n is a sphere.*

Proof. As M has one elliptic point, S must be a positive constant and H is positive somewhere. From (2) we see that H never vanishes, and so it is positive everywhere. Hence, we can write (2) as

$$\sqrt{S} \leqslant \sqrt{n(n-1)}\, H. \tag{6}$$

Integrating on M,

$$\sqrt{S}\,A \leqslant \sqrt{n(n-1)} \int_M H\,dA,$$

and taking squares we obtain

$$SA^2 \leqslant n(n-1)\left(\int_M H\,dA\right)^2, \tag{7}$$

the equality holding if and only if M is umbilical at every point.

From (3) and (4) we have

$$\begin{aligned} 0 &= n(n-1)\int_M H\,dA + \int_M S\langle \psi, N\rangle\,dA \\ &= n(n-1)\int_M H\,dA + S\int_M \langle \psi, N\rangle\,dA \\ &= n(n-1)\int_M H\,dA - (n+1)SV, \end{aligned}$$

that is,

$$\int_M H\,dA = \frac{n+1}{n(n-1)} SV. \tag{8}$$

Combining (7) and (8) we have

$$SA^2 \leqslant n(n-1)\left[\frac{n+1}{n(n-1)} SV\right]^2,$$

and so,

$$\frac{n(n-1)}{(n+1)^2}\frac{A^2}{V^2} \leqslant S. \tag{9}$$

Moreover the equality holds if and only if M is a sphere in $\mathbf{R}^{n+1}$.

Now, we will prove the opposite inequality. Let f be the solution of the Dirichlet problem such that

$$\bar{\Delta} f = 1 \text{ on } \Omega \quad \text{and} \quad z = 0 \text{ on } M.$$

From the divergence theorem,

$$V = \int_\Omega \bar{\Delta} f\,dV = -\int_M u\,dA. \tag{10}$$

The Schwarz inequality $(\bar{\Delta} f)^2 \leqslant (n+1)|\bar{\nabla}^2 f|^2$ and Reilly's formula give

$$\frac{V}{n+1} \geqslant \int_M Hu^2\, dA. \tag{11}$$

From (6), Schwarz inequality for the function u, and (10) we have

$$\int_M Hu^2\, dA \geqslant \frac{\sqrt{S}}{\sqrt{n(n-1)}} \int_M u^2\, dA \geqslant \frac{\sqrt{S}}{\sqrt{n(n-1)}\, A}\left[\int_M u\, dA\right]^2$$
$$= \frac{\sqrt{S}}{\sqrt{n(n-1)}}\frac{V^2}{A}.$$

Putting this inequality in (11) we obtain

$$\frac{V}{n+1} \geqslant \frac{\sqrt{S}}{\sqrt{n(n-1)}}\frac{V^2}{A},$$

and taking squares

$$\frac{n(n-1)}{(n+1)^2}\frac{A^2}{V^2} \geqslant S.$$

So we have the equality in (9) and the theorem is proved.

3. A congruence theorem for hypersurfaces

Theorem 2. *Let $\psi: M^n \to \mathbf{R}^{n+1}$ be a compact hypersurface embedded in the Euclidean space. Suppose that the mean curvature of ψ, with respect to the interior normal H, is nonnegative.*

(A) *If $\psi': M^n \to \mathbf{R}^m$ is a smooth map such that*

$$\int_M \sigma(\nabla\psi', \nabla\psi')\, dA \geqslant n\int_M H\, dA, \tag{12}$$

and $|\Delta\psi'| \leqslant nH$ everywhere, then ψ' differs from ψ in a rigid motion.

(B) *If $\psi': M^n \to \mathbf{R}^{n+m}$ is an isometric immersion with mean curvature vector H', satisfying $|H'| \leqslant H$ everywhere, then ψ and ψ' are congruent.*

Proof. The assertion in (B) is a trivial consequence of (A). Now we prove (A). Let $F: \bar{\Omega} \to \mathbf{R}^m$ be the solution of the Dirichlet problem

$$\bar{\Delta} F = 0 \text{ on } \Omega \quad \text{and} \quad F = \psi' \text{ in } M = \partial\Omega.$$

We put $U = \partial F/\partial N: M \to \mathbf{R}^m$. Estimating the length of the Hessian of F by zero, Rielly's formula gives

$$0 \geqslant \int_M \left[-2\langle \Delta\psi', U\rangle + nH|U|^2 + \sigma(\nabla\psi', \nabla\psi')\right] dA.$$

On the other hand, by hypothesis $\langle \Delta\psi', U\rangle \leqslant |U||\Delta\psi'| \leqslant n|U|H$, which combined with (12) transforms the above integral inequality into

$$0 \geqslant n\int_M H\left(-2|U| + |U|^2 + 1\right) dA.$$

As the integrand is clearly nonnegative, we have in fact an equality and the following consequences follow:

(i) The Hessian of F vanishes, or equivalently $F(x) = Bx + b$, for any x in $\overline{\Omega}$, B being a linear map from $\mathbf{R}^{n+1}$ to $\mathbf{R}^m$ and b a vector in $\mathbf{R}^m$. In particular $\psi' = B\psi + b$ is contained in an $(n+1)$-dimensional linear subspace of $\mathbf{R}^m$, $\psi' : M^n \to \mathbf{R}^{n+1}$.

(ii) $U = BN$, and if $H > 0$ at some point, then $|U| = 1$, i.e., $|BN| = 1$. But the set of normal directions corresponding to elliptic points of M is dense in the sphere $S^n(1)$, so $|BN| = 1$ for every unit vector N in $S^n(1)$. Hence B is orthogonal and the result is proved.

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